Chapter 1 Introduction

1.1 Background

The mathematical representation of physical entities

Three of the more important mathematical systems for representing the entities of contemporary engineering and physical science are the (three-dimensional) vector algebra, the more general tensor algebra, and geometric algebra. Grassmann algebra is more general than vector algebra, overlaps aspects of the tensor algebra, and underpins geometric algebra. It predates all three. In this book we will show that it is only via Grassmann algebra that many of the geometric and physical entities commonly used in the engineering and physical sciences may be represented mathematically in a way which correctly models their pertinent properties and leads straightforwardly to principal results.

As a case in point we may take the concept of *force*. It is well known that a force is not satisfactorily represented by a (free) vector, yet contemporary practice is still to use a (free) vector calculus for this task. The deficiency may be made up for by verbal appendages to the mathematical statements: for example 'where the force f acts along the line through the point P'. Such verbal appendages, being necessary, and yet not part of the calculus being used, indicate that the calculus itself is not adequate to model force satisfactorily. In practice this inadequacy is coped with in terms of a (free) vector calculus by the introduction of the concept of *moment*. The conditions of equilibrium of a rigid body include a condition on the sum of the moments of the forces about any point. The justification for this condition is not well treated in contemporary texts. It will be shown later however that by representing a force correctly in terms of an element of the Grassmann algebra, both force-vector and moment conditions for the equilibrium of a rigid body may be united in one condition, a natural consequence of the algebraic processes alone.

Since the application of Grassmann algebra to mechanics was known during the nineteenth century one might wonder why, with the 'progress of science', it is not currently used. Indeed the same question might be asked with respect to its application in many other fields. To attempt to answer these questions, a brief biography of Grassmann is included as an appendix. In brief, the scientific world was probably not ready in the nineteenth century for the new ideas that Grassmann proposed, and now, in the twenty-first century, seems only just becoming aware of their potential.

The central concept of the *Ausdehnungslehre*

Grassmann's principal contribution to the physical sciences was his discovery of a natural language of geometry from which he derived a geometric calculus of significant power. For a mathematical representation of a physical phenomenon to be 'correct' it must be of a tensorial nature and since many 'physical' tensors have direct geometric counterparts, a calculus applicable to geometry may be expected to find application in the physical sciences.

The word '*Ausdehnungslehre*' is most commonly translated as '*theory of extension*', the fundamental product operation of the theory then becoming known as the *exterior product*. The notion of *extension* has its roots in the interpretation of the algebra in geometric terms: an element of the algebra may be 'extended' to form a higher order element by its (exterior) product with another, in the way that a point may be extended to a line, or a line to a plane, by a point exterior to it. The notion of *exteriorness* is equivalent algebraically to that of linear independence. If the exterior product of elements of grade 1 (for example, points or vectors) is non-zero, then they are independent.

A line may be defined by the exterior product of *any* two distinct points on it. Similarly, a plane may be defined by the exterior product of *any* three distinct points in it, and so on for higher dimensions. This independence with respect to the specific points chosen is an important and fundamental property of the exterior product. Each time a higher dimensional object is required it is simply created out of a lower dimensional one by multiplying by a new element in a new dimension. Intersections of elements are also obtainable as products.

Simple elements of the Grassmann algebra may be interpreted as defining subspaces of a linear space. The exterior product then becomes the operation for building higher dimensional subspaces (higher order elements) from a set of lower dimensional independent subspaces. A second product operation called the *regressive product* may then be defined for determining the common lower dimensional subspaces of a set of higher dimensional non-independent subspaces.

Comparison with the vector and tensor algebras

The Grassmann algebra is a tensorial algebra, that is, it concerns itself with the types of mathematical entities and operations necessary to describe physical quantities in an invariant manner. In fact, it has much in common with the algebra of anti-symmetric tensors – the exterior product being equivalent to the anti-symmetric tensor product. Nevertheless, there are conceptual and notational differences which make the Grassmann algebra richer and easier to use.

Rather than a sub-algebra of the tensor algebra, it is perhaps more meaningful to view the Grassmann algebra as a super-algebra of the three-dimensional vector algebra since both commonly use invariant (coordinate-free) notations. The principal differences are that the Grassmann algebra has a dual axiomatic structure, can treat higher order elements than vectors, can differentiate between points and vectors, generalizes the notion of 'cross product', is independent of dimension, and possesses the structure of a true algebra.

Algebraicizing the notion of linear dependence

Another way of viewing Grassmann algebra is as linear or vector algebra onto which has been introduced a product operation which algebraicizes the notion of linear dependence. This product operation is called the *exterior product* and is symbolized with a wedge $\wedge$.

If vectors $x_1, x_2, x_3, \ldots$ are linearly dependent, then it turns out that their exterior product is zero: $x_1 \wedge x_2 \wedge x_3 \wedge \ldots = 0$. If they are independent, their exterior product is non-zero.

Conversely, if the exterior product of vectors $x_1, x_2, x_3, \ldots$ is zero, then the vectors are linearly dependent. Thus the exterior product brings the critical notion of linear dependence into the realm of direct algebraic manipulation.

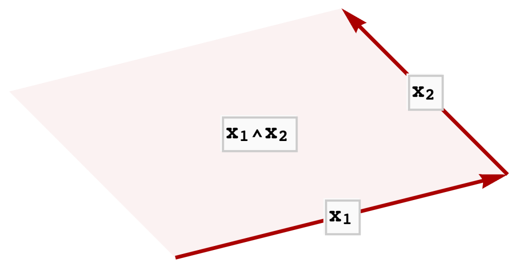
Although this might appear to be a relatively minor addition to linear algebra, we expect to demonstrate in this book that nothing could be further from the truth: the consequences of being able to model linear dependence with a product operation are far reaching, both in facilitating an understanding of current results, and in the generation of new results for many of the algebras and their entities used in science and engineering today. These include of course linear and multilinear algebra, but also vector and tensor algebra, screw algebra, hypercomplex algebras, and Clifford algebras.

Grassmann algebra as a geometric calculus

Most importantly however, Grassmann's contribution has enabled the operations and entities of all of these algebras *to be interpretable geometrically*, thus enabling us to bring to bear the power of geometric visualization and intuition into our algebraic manipulations.

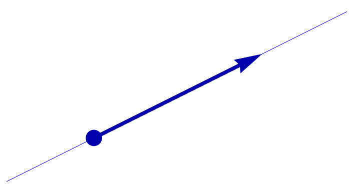
It is well known that a vector x_1 may be interpreted geometrically as representing a *direction* in space. If the space has a metric, then the *magnitude* of x_1 is interpreted as its *length*. The introduction of the exterior product enables us to *extend* the entities of the space to higher dimensions. The exterior product of two vectors $x_1 \wedge x_2$, called a *bivector*, may be visualized as the two-dimensional analogue of a direction, that is, a *planar direction*. Neither vectors nor bivectors are interpreted as being located anywhere since they do not possess sufficient information to specify independently both a direction *and* a position. If the space has a metric, then the *magnitude* of $x_1 \wedge x_2$ is interpreted as its *area*, and similarly for higher order products.

We *depict* a simple bivector by its vector factors arranged head-to-tail linked by the ghost of a parallelogram.

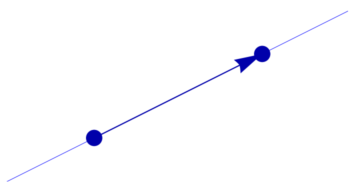


Depicting a simple bivector. This bivector is located nowhere.

For applications to the physical world, however, the Grassmann algebra possesses a critical capability that no other algebra possesses so directly: it can distinguish between *points* and *vectors* and treat them as separate entities. Lines and planes are examples of higher order constructs from points and vectors, which have both position and direction. A *line* can be represented by the exterior product of any two points on it, or by any point on it and a vector parallel to it.



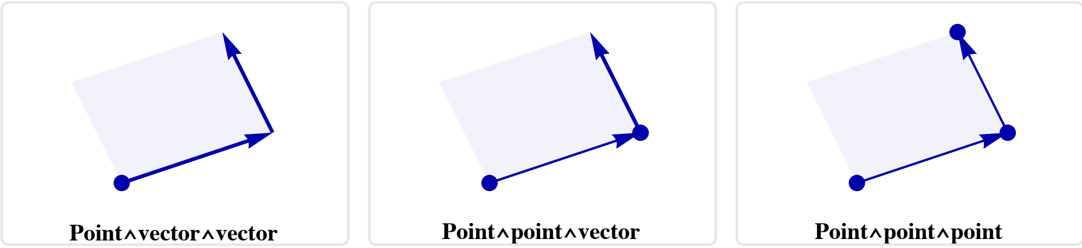
Point∧vector depiction



Point∧point depiction

Two different depictions of a bound vector in its line.

A *plane* can be represented by the exterior product of any point on it and a bivector parallel to it, any two points on it and a vector parallel to it, or any three points on it.



Three different depictions of a bound bivector.

Finally, it should be noted that the Grassmann algebra subsumes all of real algebra, the exterior product reducing in this case to the usual product operation among real numbers.

Here then is a geometric calculus *par excellence*.

1.2 The Exterior Product

The anti-symmetry of the exterior product

The exterior product of two vectors x and y of a linear space yields the bivector $x \wedge y$. The bivector is *not* a vector, and so does not belong to the original linear space. In fact the bivectors form their own linear space.

The fundamental defining characteristic of the exterior product is its *anti-symmetry*. That is, the product changes sign if the order of the factors is reversed.

$$x \wedge y == - y \wedge x \qquad 1.1$$

From this we can easily show the equivalent relation, that the exterior product of a vector with itself is zero.

$$x \wedge x == 0 \qquad 1.2$$

This is as expected because x is linearly dependent on itself.

The exterior product is associative, distributive, and behaves linearly as expected with scalars.

Exterior products of vectors in a three-dimensional space

By way of example, suppose we are working in a three-dimensional space, with basis e_1, e_2, and e_3. Then we can express vectors x and y as linear combinations of these basis vectors:

$$x == a_1 e_1 + a_2 e_2 + a_3 e_3$$

$$y == b_1 e_1 + b_2 e_2 + b_3 e_3$$

Here, the a_i and b_i are of course scalars. Taking the exterior product of x and y and multiplying out the product allows us to express the *bivector* $x \wedge y$ as a linear combination of *basis bivectors*.

$$x \wedge y == (a_1 e_1 + a_2 e_2 + a_3 e_3) \wedge (b_1 e_1 + b_2 e_2 + b_3 e_3)$$

$$x \wedge y == (a_1\, b_1)\, e_1 \wedge e_1 + (a_1\, b_2)\, e_1 \wedge e_2 + (a_1\, b_3)\, e_1 \wedge e_3 + (a_2\, b_1)\, e_2 \wedge e_1 + (a_2\, b_2)\, e_2 \wedge e_2 + (a_2\, b_3)\, e_2 \wedge e_3 + (a_3\, b_1)\, e_3 \wedge e_1 + (a_3\, b_2)\, e_3 \wedge e_2 + (a_3\, b_3)\, e_3 \wedge e_3$$

The first simplification we can make is to put all basis bivectors of the form $e_i \wedge e_i$ to zero [1.2] . The second simplification is to use the anti-symmetry of the product [1.1] and collect the terms of the bivectors which are not *essentially* different (that is, those that may differ only in the order of their factors, and hence differ only by a sign). The product $x \wedge y$ can then be written:

$$x \wedge y == (a_1\, b_2 - a_2\, b_1)\, e_1 \wedge e_2 + (a_2\, b_3 - a_3\, b_2)\, e_2 \wedge e_3 + (a_3\, b_1 - a_1\, b_3)\, e_3 \wedge e_1$$

The scalar factors appearing here are just those which would have appeared in the usual vector *cross product* of x and y. However, there is an important difference. The exterior product expression does not require the vector space to have a metric, while the usual definition of the cross product, because it generates a vector *orthogonal* to x and y, necessarily assumes a metric. Furthermore, the exterior product is associative and valid for any number of vectors in spaces of arbitrary dimension, while the cross product is not associative and is necessarily confined to products of vectors in a space of three dimensions.

For example, we may continue the product by multiplying $x \wedge y$ by a third vector z.

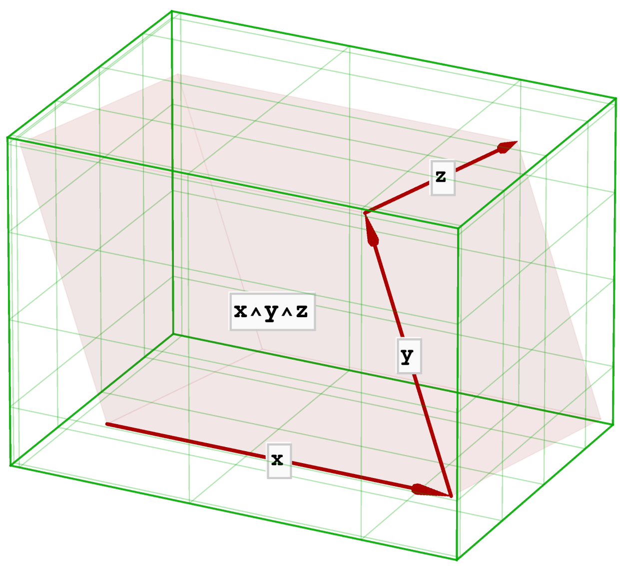
$$z == c_1\, e_1 + c_2\, e_2 + c_3\, e_3$$

$$x \wedge y \wedge z == ((a_1\, b_2 - a_2\, b_1)\, e_1 \wedge e_2 + (a_2\, b_3 - a_3\, b_2)\, e_2 \wedge e_3 + (a_3\, b_1 - a_1\, b_3)\, e_3 \wedge e_1) \wedge (c_1\, e_1 + c_2\, e_2 + c_3\, e_3)$$

Adopting the same simplification procedures as before we obtain the *trivector* $x \wedge y \wedge z$ expressed in basis form.

$$x \wedge y \wedge z == (a_1\, b_2\, c_3 - a_3\, b_2\, c_1 + a_2\, b_3\, c_1 + a_3\, b_1\, c_2 - a_1\, b_3\, c_2 - a_2\, b_1\, c_3)\, e_1 \wedge e_2 \wedge e_3$$

We *depict* a simple trivector by its vector factors arranged head-to-tail linked by the ghost of a parallelepiped.



A trivector is the exterior product of a three vectors

A trivector in a space of three dimensions has just one component. Its coefficient is the *determinant* of the coefficients of the original three vectors. Clearly, if these three vectors had been linearly dependent, this determinant would have been zero. In a metric space, this coefficient would be proportional to the *volume* of the parallelepiped formed by the vectors x, y, and z. Hence the geometric interpretation of the algebraic result: if x, y, and z are lying in a planar direction, that is, they are dependent, then the volume of the parallelepiped defined is zero.

We see here also that the exterior product begins to give geometric meaning to the often inscrutable operations of the algebra of determinants. In fact we shall see that *all* the operations of determinants are straightforward consequences of the properties of the exterior product.

In three-dimensional *metric* vector algebra, the vanishing of the scalar triple product of three vectors is often used as a criterion of their linear dependence, whereas in fact the vanishing of their exterior product (valid also in a *non-metric* space) would suffice. It is interesting to note that the notation for the scalar triple product, or 'box' product, is Grassmann's original notation for the exterior product, *viz* [x y z].

Finally, we can see that the exterior product of more than three vectors in a three-dimensional space will always be zero, since they must be dependent.

Terminology: elements and entities

To this point we have been referring to the elements of the space under discussion as *vectors*, and their higher order constructs in three dimensions as *bivectors* and *trivectors*. In the general case we will refer to the exterior product of an unspecified number of vectors as a *multivector*, and the exterior product of *m* vectors as an *m-vector*.

The word 'vector' however, is in current practice used in two distinct ways. The first and traditional use endows the vector with its well-known geometric properties of direction, sense, and (possibly) magnitude. In the second and more recent, the term vector may refer to any element of a linear space, even though the space is devoid of geometric context.

In this book, we adopt the traditional practice and use the term vector only when we intend it to have its traditional geometric interpretation of a (free) arrow-like entity. When referring to an element of a linear space which we are not specifically interpreting geometrically, we simply use the term *element*. The exterior product of *m* 1-elements of a linear space will thus be referred to as an *m-element*. (In the *GrassmannAlgebra* package however, we have had to depart somewhat from this convention in the interests of common usage: symbols representing 1-elements are called VectorSymbols).

The reason this distinction is important is because it allows us to introduce points into the algebra. By adding a new element called the *Origin* into the basis of a vector space which has the interpretation of a *point*, we are able to distinguish two types of 1-element: vectors and points. This simple distinction leads to a sophisticated and powerful tableau of free (involving only vectors) and bound (involving points) entities with which to model geometric and physical systems.

Science and engineering make use of mathematics by endowing its constructs with geometric or physical interpretations. We will use the term *entity* to refer to such a construct of elements which we specifically wish to endow with a geometric or physical interpretation. For example we would say that (geometric) *points* and *vectors* and (physical) *positions* and *directions* are 1-entities, while (geometric) *bound vectors*, *bivectors* and *screws*, and (physical) *forces* and *angular momenta* are 2-entities. Points, vectors, bound vectors and bivectors and screws are examples of *geometric* entities. Positions, directions, forces and momenta are examples of *physical* entities.

Points, lines, planes and multiplanes may also be conveniently considered for computational purposes *as* geometric entities, or we may also define them in the more common way as a set of points: the set of all points *in* the entity. (A 1-element is in an *m*-element if their exterior product is zero.) We call such a set of points a *geometric object*. We *interpret* elements of a linear space geometrically or physically, while we *represent* geometric or physical entities by elements of a linear space.

An entity need not have a unique grade. For example, we will see in Volume 2 that hypercomplex entities like complex numbers, quaternions, and Clifford numbers are usually multigraded, a typical case being the sum of a scalar (of grade 0) and a bivector (of grade 2), with a typical interpretation being that of a *rotation*.

A more complete summary of terminology is given at the end of the book.

The grade of an element

The exterior product of *m* 1-elements is called an *m*-element. The value *m* is called the *grade* of the *m*-element. For example the element $u \wedge v \wedge x \wedge y$ is of grade 4.

An *m*-element may be *denoted* by a symbol underscripted with the value *m*. For example:

$$\underset{4}{\alpha} == u \wedge v \wedge x \wedge y$$

For simplicity, however, we do not generally denote 1-elements with an underscripted '1'.

The grade of a scalar is 0. We shall see that this is a natural consequence of the exterior product axioms formulated for elements of general grade.

The *dimension* of the underlying linear space of 1-elements is denoted by *n*. Elements of grade greater than *n* are zero.

The *complementary grade* of an *m*-element in an *n*-space is *n–m*.

GrassmannAlgebra recognizes the symbol `Dimension` as the numerical dimension of the current underlying linear space; and the symbol `★n` as the symbolic dimension of any underlying linear space under theoretical consideration.

Interchanging the order of the factors in an exterior product

The exterior product is defined to be associative. Hence we can isolate any two adjacent 1-element factors. Interchanging the order of these factors will change the sign of the product:

$$\ldots \wedge x \wedge y \wedge \ldots == \ldots \wedge (x \wedge y) \wedge \ldots == \ldots \wedge (-(y \wedge x)) \wedge \ldots == -(\ldots \wedge y \wedge x \wedge \ldots)$$

In fact, interchanging the order of *any* two 1-element factors will also change the sign of the product.

$$\ldots \wedge x \wedge \ldots \wedge y \wedge \ldots == -(\ldots \wedge y \wedge \ldots \wedge x \wedge \ldots)$$

To see why this is so, suppose the number of factors between x and y is *m*. First move y to the immediate left of x. This will cause *m*+1 changes of sign. Then move x to the position that y vacated. This will cause *m* changes of sign. In all there will be 2*m*+1 changes of sign, equivalent to just one sign change.

Note that it is only elements of odd grade that anti-commute. If, in a product of two elements, at least one of them is of even grade, then the elements commute. For example, 2-elements commute with all other elements.

$$(x \wedge y) \wedge z == z \wedge (x \wedge y)$$

A brief summary of the properties of the exterior product

In this section we summarize a few of the more important properties of the exterior product some of which we have already introduced informally. In Chapter 2: The Exterior Product, the complete set of axioms is discussed.

- The exterior product of an *m*-element and a *k*-element is an (*m*+*k*)-element.

- The exterior product is associative.

$$\left(\underset{m}{\alpha} \wedge \underset{k}{\beta}\right) \wedge \underset{r}{\gamma} == \underset{m}{\alpha} \wedge \left(\underset{k}{\beta} \wedge \underset{r}{\gamma}\right) \tag{1.3}$$

- The unit scalar acts as an identity under the exterior product.

$$\underset{m}{\alpha} == 1 \wedge \underset{m}{\alpha} == \underset{m}{\alpha} \wedge 1 \tag{1.4}$$

- Scalars factor out of products.

$$\left(a\, \underset{m}{\alpha}\right) \wedge \underset{k}{\beta} == \underset{m}{\alpha} \wedge \left(a\, \underset{k}{\beta}\right) == a \left(\underset{m}{\alpha} \wedge \underset{k}{\beta}\right) \tag{1.5}$$

- An exterior product is anti-commutative whenever the grades of the factors are both odd.

$$\underset{m}{\alpha} \wedge \underset{k}{\beta} == (-1)^{m\,k}\, \underset{k}{\beta} \wedge \underset{m}{\alpha} \tag{1.6}$$

- The exterior product is both left and right distributive under addition.

$$\left(\underset{m}{\alpha} + \underset{m}{\beta}\right) \wedge \underset{r}{\gamma} == \underset{m}{\alpha} \wedge \underset{r}{\gamma} + \underset{m}{\beta} \wedge \underset{r}{\gamma} \qquad \underset{m}{\alpha} \wedge \left(\underset{r}{\beta} + \underset{r}{\gamma}\right) == \underset{m}{\alpha} \wedge \underset{r}{\beta} + \underset{m}{\alpha} \wedge \underset{r}{\gamma} \tag{1.7}$$

1.3 The Regressive Product

The regressive product as a dual product to the exterior product

One of Grassmann's major contributions, which appears to be all but lost to current mathematics, is the *regressive product*. The regressive product is the real foundation for the theory of the inner and scalar products (and their generalization, the interior product). Yet the regressive product is often ignored and the inner product defined as a new construct independent of the regressive product. This approach not only has potential for inconsistencies, but also fails to capitalize on the wealth of results available from the natural duality between the exterior and regressive products. The approach adopted in this book follows Grassmann's original concept. The regressive product is a simple dual operation to the exterior product and an enticing and powerful symmetry is lost by ignoring it, particularly in the development of metric results involving complements and interior products.

The underlying beauty of the *Ausdehnungslehre* is due to this symmetry, which in turn is due to the fact that linear spaces of *m*-elements and linear spaces of (*n*–*m*)-elements have the same dimension. This too is the key to the duality of the exterior and regressive products. For example, the exterior product of *m* 1-elements is an *m*-element. The dual to this is that the regressive product of *m* (*n*–1)-elements is an (*n*–*m*)-element. This duality has the same form as that in a Boolean algebra: if the exterior product corresponds to a type of 'union' then the regressive product corresponds to a type of 'intersection'.

It is this duality that permits the definition of complement in Chapter 5, and hence to the definition of the interior, inner and scalar products in Chapter 6. To underscore this duality it is proposed to adopt here the $\vee$ ('vee') for the regressive product operation. Unfortunately the now almost universal adoption of the 'wedge' for the exterior product (and hence the 'vee' for the regressive product) yields the reverse symbolic connotation to the notions of 'union' and 'intersection' in a Boolean algebra.

Unions and intersections of spaces

Consider a (non-zero) 2-element $x \wedge y$. We can test to see if any given 1-element z is in the subspace spanned by x and y by taking the exterior product of $x \wedge y$ with z and seeing if the result is zero. From this point of view, $x \wedge y$ is an element which can be used to *define* the subspace instead of the individual 1-elements x and y.

Thus we can define the *space* of $x \wedge y$ as the space of all 1-elements z such that $x \wedge y \wedge z = 0$. We extend this to more general elements by defining the space of a simple *m*-element A as the space of all 1-elements z such that $A \wedge z = 0$. (We discuss the notion of space in more detail in the section on terminology at the end of the book).

We will also need the notion of congruence. We will say that two elements (of any grade) are *congruent* if one is a scalar multiple of the other. For example x and $2\,x$ are congruent; $x \wedge y$ and $-x \wedge y$ are congruent. Congruent elements define the same subspace. We denote congruence by the symbol $\equiv$. The following concepts of union and intersection only make sense up to congruence.

A *union of elements* is an element defining the subspace they *together* span.

The dual concept to union of elements is *intersection of elements*. An *intersection of elements* is an element defining the subspace they span *in common*.

Suppose we have three independent 1-elements: x, y, and z. A union of $x \wedge y$ and $y \wedge z$ is any element congruent to $x \wedge y \wedge z$. An intersection of $x \wedge y$ and $y \wedge z$ is any element congruent to y.

The computation of unions and intersections by exterior and regressive products alone is limited to some special (but important) cases as we shall see below and in Chapter 3. The computation of unions and intersections in general is best done using the notion of *span*. See the section Unions and Intersections.

A brief summary of the properties of the regressive product

In this section we summarize a few of the more important properties of the regressive product. In Chapter 3: The Regressive Product, we develop the complete set of axioms from those of the exterior product. By comparing the axioms below with those for the exterior product in the previous section, we see that they are effectively generated by replacing $\wedge$ with $\vee$, and *m* by *n*–

m. The unit element 1 in its form $\underset{0}{1}$ becomes $\underset{n}{1}$.

- The regressive product of an m-element and a k-element in an n-space is an (m+k–n)-element.
- The regressive product is associative.

$$\left(\underset{m}{\alpha} \vee \underset{k}{\beta}\right) \vee \underset{r}{\gamma} == \underset{m}{\alpha} \vee \left(\underset{k}{\beta} \vee \underset{r}{\gamma}\right) \qquad 1.8$$

- The unit n-element $\underset{n}{1}$ acts as an identity under the regressive product.

$$\underset{m}{\alpha} == \underset{n}{1} \vee \underset{m}{\alpha} == \underset{m}{\alpha} \vee \underset{n}{1} \qquad 1.9$$

- Scalars factor out of products.

$$\left(a \underset{m}{\alpha}\right) \vee \underset{k}{\beta} == \underset{m}{\alpha} \vee \left(a \underset{k}{\beta}\right) == a \left(\underset{m}{\alpha} \vee \underset{k}{\beta}\right) \qquad 1.10$$

- A regressive product is anti-commutative whenever the complementary grades of the factors are both odd.

$$\underset{m}{\alpha} \vee \underset{k}{\beta} == (-1)^{(n-m)\,(n-k)} \underset{k}{\beta} \vee \underset{m}{\alpha} \qquad 1.11$$

- The regressive product is both left and right distributive under addition.

$$\left(\underset{m}{\alpha} + \underset{m}{\beta}\right) \vee \underset{r}{\gamma} == \underset{m}{\alpha} \vee \underset{r}{\gamma} + \underset{m}{\beta} \vee \underset{r}{\gamma} \qquad \underset{m}{\alpha} \vee \left(\underset{r}{\beta} + \underset{r}{\gamma}\right) == \underset{m}{\alpha} \vee \underset{r}{\beta} + \underset{m}{\alpha} \vee \underset{r}{\gamma} \qquad 1.12$$

Note that when using the *GrassmannAlgebra* application, the unit n-element should be denoted by $\underset{\star n}{1}$ to ensure a correct interpretation of the symbol.

The Common Factor Axiom

Up to this point we have no way of connecting the dual axiom structures of the exterior and regressive products. That is, given a regressive product of an m-element and a k-element, how do we find the (m+k–n)-element to which it is equivalent, expressed only in terms of exterior products?

To make this connection we need to introduce a further axiom which we call the Common Factor Axiom. The form of the Common Factor Axiom may seem somewhat arbitrary, but it is in fact one of the simplest forms which enable intersections to be calculated. This can be seen in the following application of the axiom to a vector 3-space.

Suppose x, y, and z are three independent vectors in a vector 3-space. The Common Factor Axiom says that the regressive product of the two bivectors $x \wedge z$ and $y \wedge z$ may also be expressed as the regressive product of the trivector $x \wedge y \wedge z$ with their common factor z.

$$(x \wedge z) \vee (y \wedge z) == (x \wedge y \wedge z) \vee z$$

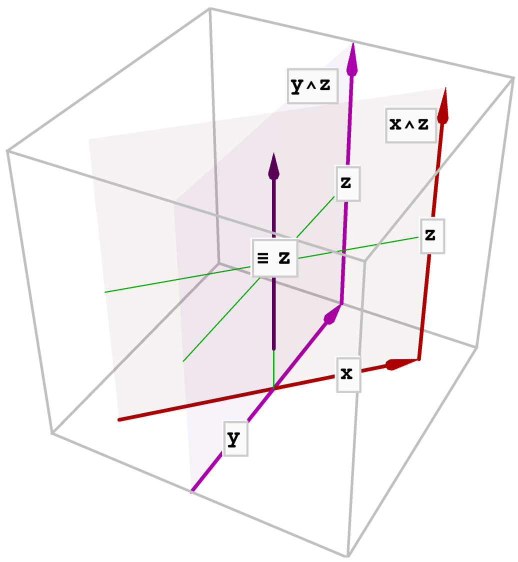
Since the space is 3-dimensional, we can write any trivector such as $x \wedge y \wedge z$ as a scalar factor (a, say) times the unit trivector (introduced in axiom 1.9).

$$(x \wedge y \wedge z) \vee z == \left(a\, \underset{3}{1}\right) \vee z == a\, z$$

This then gives us the axiomatic structure to say that the regressive product of two such elements possessing an element in common is congruent to that element.

$$(x \wedge z) \vee (y \wedge z) \equiv z$$

We *depict* this relation by showing the common factor (intersection) of the bivectors docked in a convenient position relative to each other. Remember, the bivectors are not actually located anywhere!



The intersection of two bivectors in a three−dimensional space

Of course this is just a simple case. More generally, let $\underset{m}{\alpha}$, $\underset{k}{\beta}$, and $\underset{s}{\mu}$ be *simple* elements with *m*+*k*+*s* = *n*, where *n* is the dimension of the space. Then the *Common Factor Axiom* states that

$$\left(\underset{m}{\alpha} \wedge \underset{s}{\mu}\right) \vee \left(\underset{k}{\beta} \wedge \underset{s}{\mu}\right) == \left(\underset{m}{\alpha} \wedge \underset{k}{\beta} \wedge \underset{s}{\mu}\right) \vee \underset{s}{\mu} \qquad m + k + s == n \tag{1.13}$$

There are many rearrangements and special cases of this formula which we will encounter in later chapters. For example, when *s* is zero, the Common Factor Axiom shows that the regressive product of an *m*-element with an (*n*–*m*)-element is a scalar which can be expressed in the alternative form of a regressive product with the unit 1.

$$\underset{m}{\alpha} \vee \underset{n-m}{\beta} == \left(\underset{m}{\alpha} \wedge \underset{n-m}{\beta}\right) \vee 1$$

The Common Factor Axiom allows us to prove a particularly useful result: the Common Factor Theorem. The Common Factor Theorem expresses *any* regressive product in terms of exterior products alone. This of course enables us to calculate intersections of more general elements.

Most importantly however we will see later that the Common Factor Theorem has a counterpart expressed in terms of exterior and interior products, called the Interior Common Factor Theorem. This forms the principal expansion theorem for interior products and from which we can derive many of the most important theorems relating exterior and interior products.

The Interior Common Factor Theorem, and the Common Factor Theorem upon which it is based, are possibly the most important theorems in the Grassmann algebra.

In the next section we informally apply the Common Factor Theorem to obtain the intersection of two bivectors in a three-dimensional space.

The intersection of two bivectors in a three-dimensional space

Suppose that $x \wedge y$ and $u \wedge v$ are non-congruent bivectors in a three dimensional space. Since the space has only three dimensions, the bivectors must have an intersection. We denote the regressive product of $x \wedge y$ and $u \wedge v$ by z:

$$z == (x \wedge y) \vee (u \wedge v)$$

We will see in Chapter 3: The Regressive Product that this can be expanded by the *Common Factor Theorem* to give

$$(x \wedge y) \vee (u \wedge v) == (x \wedge y \wedge v) \vee u - (x \wedge y \wedge u) \vee v \qquad 1.14$$

But we have already seen in section 1.2 that in a 3-space, the exterior product of three vectors will, in any given basis, give the basis trivector, multiplied by the determinant of the components of the vectors making up the trivector.

Additionally, we note that the regressive product (intersection) of a vector with an element like the basis trivector completely containing the vector, will just give an element congruent to itself. Thus the regressive product leads us to an explicit expression congruent to the intersection of the two bivectors.

$$z \equiv \text{Det}[x, y, v]\, u - \text{Det}[x, y, u]\, v$$

Here `Det[x,y,v]` is the determinant of the components of x, y, and v in the chosen basis. We could also have obtained an equivalent formula expressing z in terms of x and y instead of u and v by simply interchanging the order of the bivector factors in the original regressive product.

Note carefully however, that *this formula only finds the common factor up to congruence*, because until we determine an explicit expression for the unit *n*-element in terms of basis elements (which we do by introducing the complement operation in Chapter 5), we cannot usefully use axiom [1.9] above. Nevertheless, this is not to be seen as a restriction. Rather, as we shall see in the next section it leads to interesting insights as to what can be accomplished when we work in spaces without a metric, such as projective spaces.

1.4 Geometric Interpretations

Points and vectors

In this section we introduce *two* different types of *geometrically interpreted* elements which can be represented by the elements of a linear space: *vectors* and *points*. Then we look at the interpretations of the various higher grade elements that we can generate from them by the exterior product. Finally we see how the regressive product can be used to calculate intersections of these higher order elements.

As discussed in section 1.2, the term ‘vector’ is often used to refer to an element of a linear space with no intention of implying an interpretation. In this book however, we reserve the term for a particular type of geometric interpretation: that associated with representing *direction*. Exterior products of vectors then represent higher-dimensional analogues to the notion of direction.

But an element of a linear space may also be interpreted as a *point*. Of course vectors may also be used to represent points, but only *relative to another given point*. Hence they cannot represent absolute position. These vectors are properly called *position vectors*. Common practice often omits explicit reference to this other given point, or perhaps may refer to it verbally. Points can be represented satisfactorily in many cases by position vectors alone, but when *both position and direction* are required in the same element we must distinguish mathematically between the two.

To describe true position in a three-dimensional physical space, a linear space of *four* dimensions is required, one for an origin point, and the other three for the three spatial directions. Since the exterior product is independent of the dimension of the underlying space, it can deal satisfactorily with points and vectors together. The usual three-dimensional vector algebra however cannot.

Suppose x, y, and z are elements of a linear space interpreted as vectors. Vectors always have a direction. But only when the linear space has a metric do they also have a *magnitude*. Since to this stage we have not yet introduced the notion of metric, we will only be discussing interpretations and applications which do not require elements (other than congruent elements) to be commensurable.

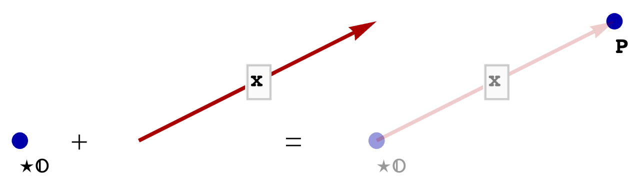
Of course, vectors may be summed in a space with no metric, the standard geometric interpretation of this operation being either the ‘triangle rule’ or the ‘parallelogram rule’.

Sums and differences of points

A *point* is defined as the sum of the *origin point* and a *vector*. If ★O is the origin, and x is a vector, then ★O + x is a point.

$$P == \star\mathbb{O} + x \qquad 1.15$$

The vector x is called the *position vector* of the point P.

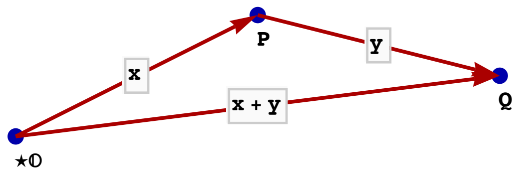


The sum of the origin and a vector is a point

The *sum of a point and a vector* is another point.

$$Q == P + y == (\star\mathbb{O} + x) + y == \star\mathbb{O} + (x + y)$$

We depict this by conveniently docking the tails of the vectors at the points.

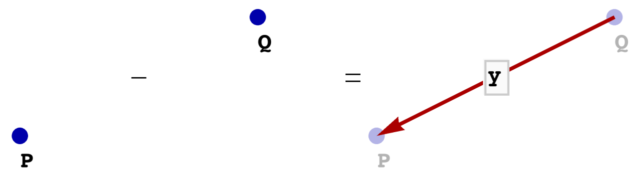


The sum of a point and a vector is another point

The *difference of two points* is a vector since the origins cancel.

$$P - Q == (\star\mathbb{O} + x) - (\star\mathbb{O} + x + y) == y$$

This simple result is actually the seminal underpinning of the relationship between points and vectors.



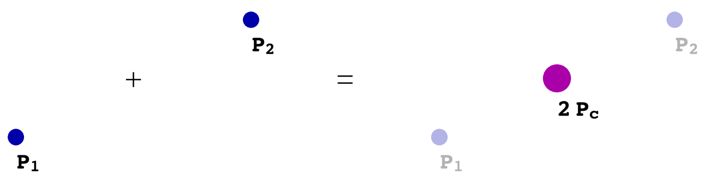
The difference of two points is a vector

A scalar multiple of a point is called a *weighted point*. For example, if `m` is a scalar, `m P` is a weighted point with weight `m`.

The *sum* of two points gives the point halfway between them with a weight of 2.

$$P_1 + P_2 == (\star\mathbb{O} + x_1) + (\star\mathbb{O} + x_2) == 2\left(\star\mathbb{O} + \frac{x_1 + x_2}{2}\right) == 2\, P_c$$

Thus the sum of two points yields a result which is of quite a different nature to their difference. The sum of two points is a weighted point, while their difference is a vector - the first a located entity; the second un-located.



The sum of two points is a point of weight 2 located mid−way between them

◆ ***Historical Note***

The point was originally considered the fundamental geometric entity of interest. However the difference of points was clearly no longer a point, since reference to the origin had been lost. Sir William Rowan Hamilton coined the term ‘vector’ for this new entity since adding a vector to a point ‘carried’ the point to a new point.

Determining a mass-centre

A classic application of a sum of weighted points is to the determination of a centre of mass.

Consider a collection of points P_i weighted with masses m_i. The sum of the weighted points gives the point P_G at the mass-centre (centre of gravity) weighted with the total mass M.

To show this, first add the weighted points and collect the terms involving the origin.

$$M P_G == m_1 (\star\mathbb{O} + x_1) + m_2 (\star\mathbb{O} + x_2) + m_3 (\star\mathbb{O} + x_3) + \ldots$$
$$== (m_1 + m_2 + m_3 + \ldots) \star\mathbb{O} + (m_1 x_1 + m_2 x_2 + m_3 x_3 + \ldots)$$

Dividing through by the total mass M gives the centre of mass.

$$P_G == \star\mathbb{O} + \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 + \ldots}{m_1 + m_2 + m_3 + \ldots}$$

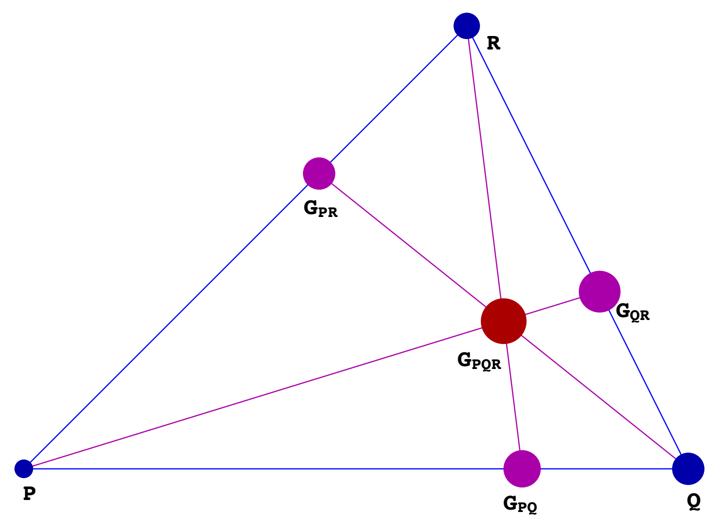
- To fix ideas, we take a simple example demonstrating that centres of mass can be accumulated in any order. Suppose we have three points P, Q, and R with masses p, q, and r. The centres of mass taken two at a time are given by

$$(p + q) G_{PQ} == p P + q Q$$
$$(q + r) G_{QR} == q Q + r R$$
$$(p + r) G_{PR} == p P + r R$$

Now take the centre of mass of each of these with the other weighted point. Clearly, the three sums will be equal.

$$(p + q) G_{PQ} + r R == (q + r) G_{QR} + p P == (p + r) G_{PR} + q Q$$
$$== p P + q Q + r R == (p + q + r) G_{PQR}$$

It is straightforward to depict these relationships. In the diagram below we have depicted the mass of a point by its area.



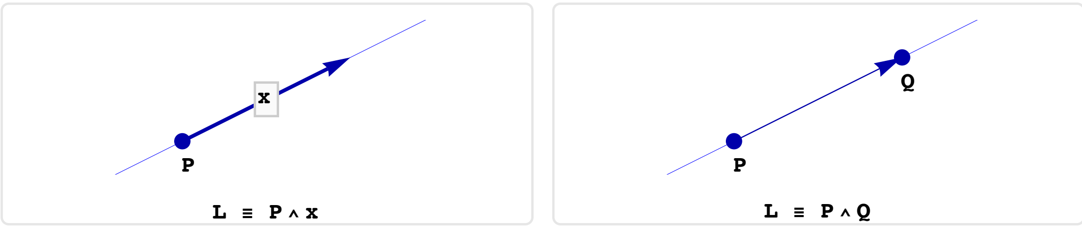
Centres of mass can be accumulated in any order

Lines and planes

The exterior product of a point and a vector gives a *bound vector*. Bound vectors are the entities we need for mathematically representing lines. A line is the set of points *in* the bound vector. That is, it consists of all the points whose exterior product with the bound vector is zero.

In practice, we usually *compute* with lines by computing with their bound vectors. For example, to get the intersection of two lines in the plane, we take the regressive product of their bound vectors. By abuse of terminology we may therefore often refer to a bound vector *as* a line, or to a line as a bound vector.

A bound vector can be defined by the exterior product of a point and a vector, or of two points. In the first case we represent the line L *through* the point P in the direction of x by any entity *congruent* to the exterior product of P and x. In the second case we can introduce Q as $P + x$ to get the same result.



Two different depictions of a bound vector in its line.

$$L \equiv P \wedge x \equiv P \wedge (Q - P) \equiv P \wedge Q$$

A line is independent of the specific point used to define it. To see this, consider any other point R on the line. Since R is on the line it can be represented by the sum of P with an arbitrary scalar multiple of the vector x:

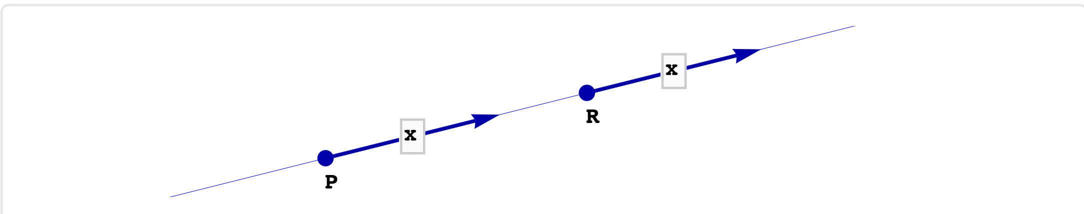
$$L \equiv R \wedge x == (P + a\, x) \wedge x == P \wedge x$$

A line may also be represented by the exterior product of *any two points* on it.

$$L \equiv P \wedge R == P \wedge (P + a\, x) == a\, P \wedge x$$

Note that the bound vectors $P \wedge x$ and $P \wedge R$ are (in general) different, but congruent. They therefore define the same line.

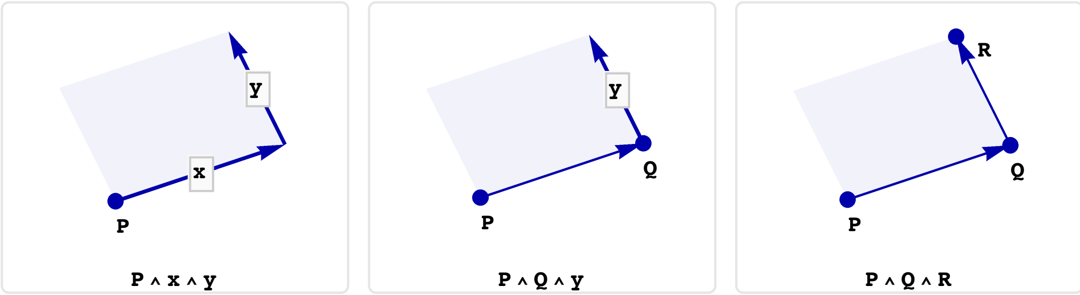
$$L \equiv P \wedge x \equiv R \wedge x \equiv P \wedge R \qquad 1.16$$



Two bound vectors defining the same line

These concepts extend naturally to higher dimensional constructs. For example a plane Π may be represented by the exterior product of single point on it together with a bivector in the direction of the plane, any two points on it together with a vector in it (not parallel to the line joining the points), or any three points on it (not in the same line).

$$\Pi \equiv P \wedge x \wedge y \equiv P \wedge Q \wedge y \equiv P \wedge Q \wedge R \qquad 1.17$$



Three different depictions of a bound bivector in its plane.

To build higher dimensional geometric entities from lower dimensional ones, we simply take their exterior product. For example we can build a line by taking the exterior product of a point with any point or vector *exterior* to it. Or we can build a plane by taking the exterior product of a line with any point or vector exterior to it.

The intersection of two lines

We can use the regressive product to find the intersection of two geometric entities if together the entities span the whole space. For example, suppose we have two lines in a plane and we want to find the point of intersection P. As we have seen we can represent the lines in a number of ways. For example:

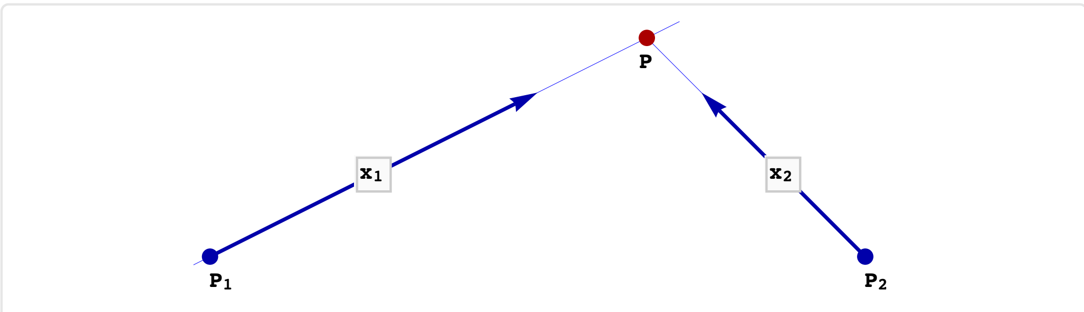
$$L_1 \equiv P_1 \wedge x_1 == (\star\mathbb{O} + \nu_1) \wedge x_1 == \star\mathbb{O} \wedge x_1 + \nu_1 \wedge x_1$$

$$L_2 \equiv P_2 \wedge x_2 == (\star\mathbb{O} + \nu_2) \wedge x_2 == \star\mathbb{O} \wedge x_2 + \nu_2 \wedge x_2$$

The point of intersection of L_1 and L_2 is the point P given by (congruent to) the regressive product of the lines L_1 and L_2.

$$P \equiv L_1 \vee L_2 == (\star\mathbb{O} \wedge x_1 + \nu_1 \wedge x_1) \vee (\star\mathbb{O} \wedge x_2 + \nu_2 \wedge x_2)$$

Here we depict the lines overlaid by their defining bound vectors.



The intersection of lines as the common factor of bound vectors

Expanding the formula for P gives four terms.

$$P \equiv (\star\mathbb{O} \wedge x_1) \vee (\star\mathbb{O} \wedge x_2) + (\nu_1 \wedge x_1) \vee (\star\mathbb{O} \wedge x_2) + (\star\mathbb{O} \wedge x_1) \vee (\nu_2 \wedge x_2) + (\nu_1 \wedge x_1) \vee (\nu_2 \wedge x_2)$$

The Common Factor Theorem for the regressive product of elements of the form $(x \wedge y) \vee (u \wedge v)$ in a linear space of three dimensions was introduced as formula [1.14] in section 1.3 as

$$(x \wedge y) \vee (u \wedge v) == (x \wedge y \wedge v) \vee u - (x \wedge y \wedge u) \vee v$$

Since a bound 2-space is three dimensional (its basis contains three elements - the origin and two vectors), we can use this formula to expand each of the terms in P.

$$(\star\mathbb{O} \wedge x_1) \vee (\star\mathbb{O} \wedge x_2) == (\star\mathbb{O} \wedge x_1 \wedge x_2) \vee \star\mathbb{O} - (\star\mathbb{O} \wedge x_1 \wedge \star\mathbb{O}) \vee x_2$$

$$(\nu_1 \wedge x_1) \vee (\star\mathbb{O} \wedge x_2) == (\nu_1 \wedge x_1 \wedge x_2) \vee \star\mathbb{O} - (\nu_1 \wedge x_1 \wedge \star\mathbb{O}) \vee x_2$$

$$(\star\mathbb{O} \wedge x_1) \vee (\nu_2 \wedge x_2) == -(\nu_2 \wedge x_2) \vee (\star\mathbb{O} \wedge x_1)$$
$$== -(\nu_2 \wedge x_2 \wedge x_1) \vee \star\mathbb{O} + (\nu_2 \wedge x_2 \wedge \star\mathbb{O}) \vee x_1$$

$$(\nu_1 \wedge x_1) \vee (\nu_2 \wedge x_2) == (\nu_1 \wedge x_1 \wedge x_2) \vee \nu_2 - (\nu_1 \wedge x_1 \wedge \nu_2) \vee x_2$$

The term $\star\mathbb{O} \wedge x_1 \wedge \star\mathbb{O}$ is zero because of the exterior product of repeated factors. The four terms involving the exterior product of three vectors, for example $\nu_1 \wedge x_1 \wedge x_2$, are also zero since any three vectors in a two-dimensional vector space must be dependent (The vector space is 2-dimensional since it is the vector sub-space of a bound 2-space). Hence we can express the point of intersection P as congruent to a weighted point.

$$P \equiv (\star\mathbb{O} \wedge x_1 \wedge x_2) \vee \star\mathbb{O} + (\star\mathbb{O} \wedge \nu_2 \wedge x_2) \vee x_1 - (\star\mathbb{O} \wedge \nu_1 \wedge x_1) \vee x_2$$

If we express the vectors in terms of a basis, e_1 and e_2 say, we can reduce this formula (after some manipulation) to:

$$P \equiv \star\mathbb{O} + \frac{\mathrm{Det}[\nu_2, x_2]}{\mathrm{Det}[x_1, x_2]} x_1 - \frac{\mathrm{Det}[\nu_1, x_1]}{\mathrm{Det}[x_1, x_2]} x_2$$

Here, the determinants are the determinants of the coefficients of the vectors in the given basis.

To verify that P does indeed lie on both the lines L_1 and L_2, we only need to carry out the straightforward verification that the products $P \wedge L_1$ and $P \wedge L_2$ are both zero.

Although this approach *in this simple case* is certainly more complex than the standard algebraic approach in the plane, its interest lies in the facts that it is immediately generalizable to intersections of any geometric objects in spaces of any number of dimensions, and that it leads to easily computable solutions.

1.5 The Complement

The complement as a correspondence between spaces

The Grassmann algebra has a duality in its structure which not only gives it a certain elegance, but is also the basis of its power. We have already introduced the regressive product as the dual *product operation* to the exterior product. In this section we extend the notion of duality to define the *complement of an element*. The notions of metric, orthogonality, and interior, inner and scalar products are all based on the complement.

Consider a linear space of dimension *n* with basis $e_1, e_2, \ldots, e_n$. The set of all the essentially different *m*-element products of these basis elements forms the basis of another linear space, but this time of dimension $\binom{n}{m}$. For example, when *n* is 3, the linear space of 2-elements has three elements in its basis: $e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3$.

The anti-symmetric nature of the exterior product means that there are just as many basis elements in the linear space of (*n*–*m*)-elements as there are in the linear space of *m*-elements. Because these linear spaces have the same dimension, we can set up a *correspondence* between *m*-elements and (*n*–*m*)-elements. That is, given any *m*-element, we can define its corresponding (*n*–*m*)-element. The (*n*–*m*)-element is called the *complement* of the *m*-element. Normally this correspondence is set up between basis elements and extended to all other elements by linearity.

The Euclidean complement

Suppose we have a three-dimensional linear space with basis e_1, e_2, e_3. We *define* the *Euclidean complement* of each of the basis elements as the basis 2-element whose exterior product with the basis element gives the basis 3-element $e_1 \wedge e_2 \wedge e_3$. We denote the complement of an element by placing a 'bar' over it. Thus:

$$\overline{e_1} == e_2 \wedge e_3 \quad \Longrightarrow \quad e_1 \wedge \overline{e_1} == e_1 \wedge e_2 \wedge e_3$$

$$\overline{e_2} == e_3 \wedge e_1 \quad \Longrightarrow \quad e_2 \wedge \overline{e_2} == e_1 \wedge e_2 \wedge e_3$$

$$\overline{e_3} == e_1 \wedge e_2 \quad \Longrightarrow \quad e_3 \wedge \overline{e_3} == e_1 \wedge e_2 \wedge e_3$$

The Euclidean complement is the simplest type of complement and defines a Euclidean metric, that is, where the basis elements are mutually orthonormal. This was the only type of complement considered by Grassmann. In Chapter 5: The Complement, we will show however, that Grassmann's concept of complement is easily extended to more general metrics. Note carefully that we will be using the notion of complement to *define* the notions of orthogonality and metric, and until we do this, we will not be relying on their existence in Chapters 2, 3, and 4.

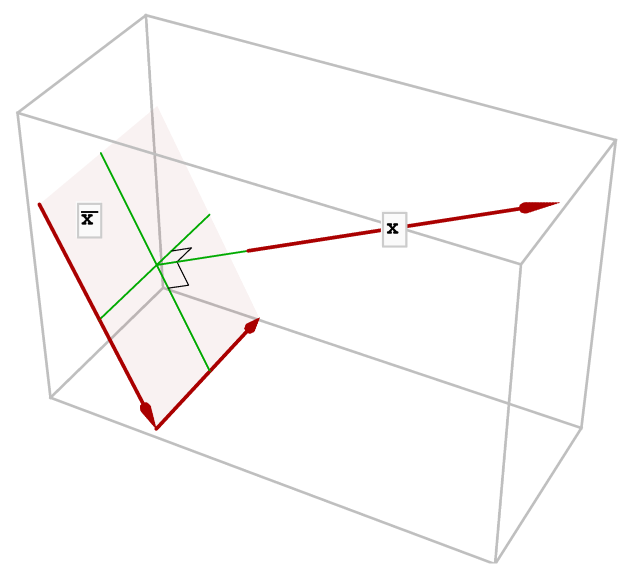
With the definitions above, we can now proceed to define the Euclidean complement of a general 1-element x in a three-dimensional space.

$$x == a\, e_1 + b\, e_2 + c\, e_3$$

To do this we need to endow the complement operation with the property of linearity so that it has meaning for linear combinations of basis elements.

$$\overline{x} == \overline{a\, e_1 + b\, e_2 + c\, e_3} == a\, \overline{e_1} + b\, \overline{e_2} + c\, \overline{e_3} == a\, e_2 \wedge e_3 + b\, e_3 \wedge e_1 + c\, e_1 \wedge e_2$$

In a vector 3-space, the complement of the vector x is the bivector $\overline{x}$, and the complement of the bivector $\overline{x}$ is the vector x. Hence they are mutual complements.



A vector and its bivector complement in a three–dimensional vector space

In section 1.6 we will see that the complement of an element is orthogonal to the element, because we will define the interior product (and hence inner and scalar products) using the complement. We can start to see how the scalar product of a 1-element with itself might arise by expanding the exterior product of x with its complement to exhibit the expected scalar as the coefficient of the basis *n*-element.

$$x \wedge \overline{x} == (a\, e_1 + b\, e_2 + c\, e_3) \wedge (a\, e_2 \wedge e_3 + b\, e_3 \wedge e_1 + c\, e_1 \wedge e_2)$$
$$== \left(a^2 + b^2 + c^2\right) e_1 \wedge e_2 \wedge e_3$$

The Euclidean complement of a basis 2-element can be defined in a manner analogous to that for 1-elements, that is, such that the exterior product of a basis 2-element with its complement is equal to the basis 3-element. The complement of a 2-element in 3-space is therefore a 1-element.

$$\overline{e_2 \wedge e_3} == e_1 \quad \Longrightarrow \quad e_2 \wedge e_3 \wedge \overline{e_2 \wedge e_3} == e_2 \wedge e_3 \wedge e_1 == e_1 \wedge e_2 \wedge e_3$$

$$\overline{e_3 \wedge e_1} == e_2 \quad \Longrightarrow \quad e_3 \wedge e_1 \wedge \overline{e_3 \wedge e_1} == e_3 \wedge e_1 \wedge e_2 == e_1 \wedge e_2 \wedge e_3$$

$$\overline{e_1 \wedge e_2} == e_3 \quad \Longrightarrow \quad e_1 \wedge e_2 \wedge \overline{e_1 \wedge e_2} == e_1 \wedge e_2 \wedge e_3$$

To complete the definition of Euclidean complement in a 3-space we note that

$$\overline{1} == e_1 \wedge e_2 \wedge e_3 \qquad \overline{e_1 \wedge e_2 \wedge e_3} == 1$$

Summarizing these results for the Euclidean complement of the basis elements of a Grassmann algebra in three dimensions shows the essential symmetry of the complement operation.

Complement Palette

Basis	Complement
1	$e_1 \wedge e_2 \wedge e_3$
e_1	$e_2 \wedge e_3$
e_2	$-(e_1 \wedge e_3)$
e_3	$e_1 \wedge e_2$
$e_1 \wedge e_2$	e_3
$e_1 \wedge e_3$	$-e_2$
$e_2 \wedge e_3$	e_1
$e_1 \wedge e_2 \wedge e_3$	1

The complement of a complement

Applying the Euclidean complement operation twice to a 1-element x shows that the complement of the complement of x in a 3-space is just x itself.

$$x == a\, e_1 + b\, e_2 + c\, e_3$$

$$\overline{x} == \overline{a\, e_1 + b\, e_2 + c\, e_3} == a\, \overline{e_1} + b\, \overline{e_2} + c\, \overline{e_3} == a\, e_2 \wedge e_3 + b\, e_3 \wedge e_1 + c\, e_1 \wedge e_2$$

$$\overline{\overline{x}} == \overline{a\, e_2 \wedge e_3 + b\, e_3 \wedge e_1 + c\, e_1 \wedge e_2} == a\, \overline{e_2 \wedge e_3} + b\, \overline{e_3 \wedge e_1} + c\, \overline{e_1 \wedge e_2}$$
$$== a\, e_1 + b\, e_2 + c\, e_3$$

$$\Longrightarrow \quad \overline{\overline{x}} == x$$

More generally, as we shall see in Chapter 5: The Complement, we can show that the complement of the complement of *any* element is the element itself, apart from a possible sign.

$$\overline{\overline{\underset{m}{\alpha}}} == (-1)^{m (n-m)} \underset{m}{\alpha} \qquad \mathbf{1.18}$$

This result is independent of the correspondence that we set up between the *m*-elements and (*n*–*m*)-elements of the space, *except that the correspondence must be symmetric*. This is equivalent to the requirement that the metric tensor (and inner product) be symmetric.

Whereas in a 3-space, the complement of the complement of a 1-element is the element itself, in a 2-space it turns out to be the negative of the element. Here is a palette of basis elements and their complements for a 2-space.

Complement Palette

Basis	Complement
1	$e_1 \wedge e_2$
e_1	e_2
e_2	$-e_1$
$e_1 \wedge e_2$	1

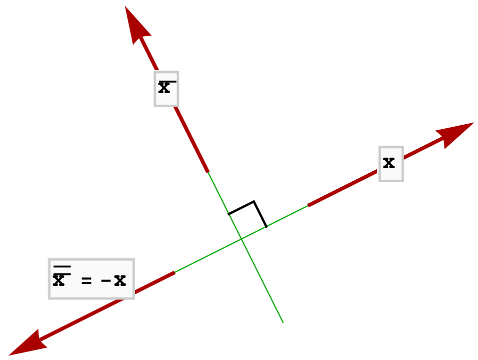
Although this sign dependence on the dimension of the space and grade of the element might appear arbitrary, it turns out to capture some essential properties of the elements in their spaces to which we have become accustomed. For example in a 2-space, taking the complement of a vector x once rotates it anticlockwise by $\frac{\pi}{2}$. Taking the complement twice rotates it anticlockwise by a further right angle into $-x$.

$$x == a\, e_1 + b\, e_2$$

$$\overline{x} == a\, \overline{e_1} + b\, \overline{e_2} == a\, e_2 - b\, e_1$$

$$\overline{\overline{x}} == a\, \overline{e_2} - b\, \overline{e_1} == -a\, e_1 - b\, e_2$$

Hence any vector and its complement can form an orthogonal basis for a vector 2-space.



The complement operation rotating vectors in a vector 2–space

The Complement Axiom

From the Common Factor Axiom we can derive a powerful relationship between the Euclidean complements of elements and their exterior and regressive products. The Euclidean complement of the exterior product of two elements is equal to the regressive product of their complements.

$$\overline{A \wedge B} == \bar{A} \vee \bar{B} \qquad 1.19$$

However, although this may be *derived* in this simple case, to develop the Grassmann algebra for general metrics, we will *assume* this relationship holds independent of the metric. It thus takes on the mantle of an axiom.

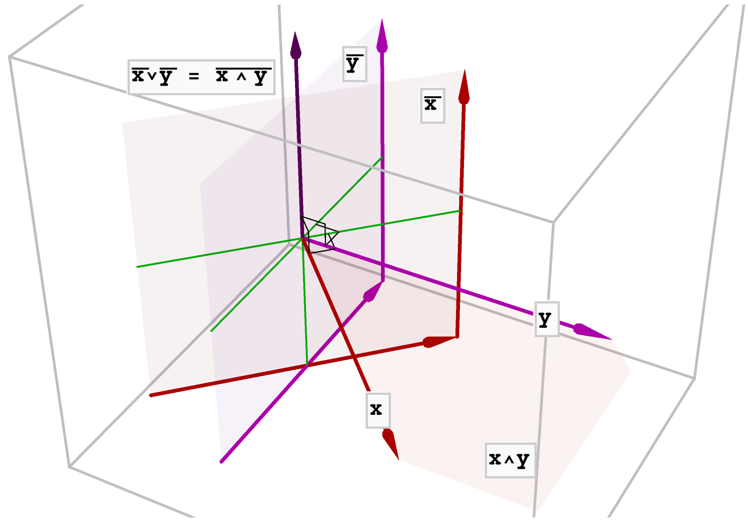
This axiom, which we call the Complement Axiom, is the quintessential formula of the Grassmann algebra. It expresses the duality of its two fundamental operations on elements and their complements. We note the formal similarity to de Morgan's law in Boolean algebra.

We will also see that adopting this formula for general complements will enable us to compute the complement of any element of a space once we have defined the complements of its basis 1-elements.

- As an example consider two vectors x and y in 3-space, and their exterior product. The Complement Axiom becomes

  $$\overline{x \wedge y} == \bar{x} \vee \bar{y}$$

The complement of the bivector $x \wedge y$ is a vector. The complements of x and y are bivectors. The regressive product of these two bivectors is a vector. The following graphic depicts this relationship, and the orthogonality of the elements. We discuss orthogonality in the next section.



Visualizing the complement axiom in vector three−space

1.6 The Interior Product

The definition of the interior product

The interior product is a generalization of the inner or scalar product to elements of arbitrary grade. First we will define the interior product and then show how the inner and scalar products are special cases.

The *interior product* of an element $\underset{m}{\alpha}$ with an element $\underset{k}{\beta}$ is denoted $\underset{m}{\alpha} \ominus \underset{k}{\beta}$ and *defined* to be the *regressive* product of $\underset{m}{\alpha}$ with the complement of $\underset{k}{\beta}$.

$$\underset{m}{\alpha} \ominus \underset{k}{\beta} == \underset{m}{\alpha} \vee \overline{\underset{k}{\beta}} \qquad 1.20$$

The grade of an interior product $\underset{m}{\alpha} \ominus \underset{k}{\beta}$ may be seen from the definition to be $m+(n-k)-n = m-k$.

Note that while the grade of a regressive product depends on the dimension of the underlying linear space, the grade of an interior product is *independent* of the dimension of the underlying space. This independence underpins the important role the interior product plays in the Grassmann algebra - the exterior product sums grades while the interior product differences them. However, grades may be arbitrarily summed, but not arbitrarily differenced, since there are no elements of negative grade.

Thus the order of factors in an interior product is important. When the grade of the first element is *less* than that of the second element, the result is necessarily zero.

Inner products and scalar products

The interior product of two elements $\underset{m}{\alpha}$ and $\underset{m}{\beta}$ *of the same grade* is (also) called their *inner* *product*. Since the grade of an interior product is the difference of the grades of its factors, an inner product is always of grade zero, hence scalar.

In the case that the two factors of the product are of grade 1, the product is called a *scalar product*. This conforms to common usage.

In Chapter 6 we will show that *the inner product is symmetric*, that is, the order of the factors is immaterial.

$$\underset{m}{\alpha} \ominus \underset{m}{\beta} == \underset{m}{\beta} \ominus \underset{m}{\alpha} \quad 1.21$$

When the inner product is between *simple* elements it can be expressed as the determinant of the array of scalar products according to the following formula:

$$(\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_m) \ominus (\beta_1 \wedge \beta_2 \wedge \cdots \wedge \beta_m) == \mathrm{Det}[\alpha_i \ominus \beta_j] \quad 1.22$$

For example, the inner product of two 2-elements $\alpha_1 \wedge \alpha_2$ and $\alpha_1 \wedge \beta_2$ may be written

$$(\alpha_1 \wedge \alpha_2) \ominus (\beta_1 \wedge \beta_2) == (\alpha_1 \ominus \beta_1)(\alpha_2 \ominus \beta_2) - (\alpha_1 \ominus \beta_2)(\alpha_2 \ominus \beta_1) \quad 1.23$$

Sequential interior products

Definition [1.20] for the interior product leads to an immediate and powerful formula relating exterior and interior products by grace of the associativity of the regressive product and the Complement Axiom [1.19].

$$\mathbf{A} \ominus (\beta_1 \wedge \beta_2 \wedge \ldots \wedge \beta_k) == \mathbf{A} \vee \overline{(\beta_1 \wedge \beta_2 \wedge \ldots \wedge \beta_k)}$$
$$== \mathbf{A} \vee \overline{\beta_1} \vee \overline{\beta_2} \vee \ldots \vee \overline{\beta_k} == (((\mathbf{A} \ominus \beta_1) \ominus \beta_2) \ominus \ldots) \ominus \beta_k$$

$$\begin{aligned}&\mathbf{A} \ominus (\beta_1 \wedge \beta_2 \wedge \ldots \wedge \beta_k)\\ &== (\mathbf{A} \ominus \beta_1) \ominus (\beta_2 \wedge \ldots \wedge \beta_k)\\ &== (((\mathbf{A} \ominus \beta_1) \ominus \beta_2) \ominus \ldots) \ominus \beta_k\end{aligned} \quad 1.24$$

By reordering the β_i factors it becomes apparent that there are many different forms in which these formulae can be expressed. For example, the inner product of two bivectors can be rewritten to display the interior product of the first bivector with either of the vectors

$$(\mathbf{x} \wedge \mathbf{y}) \ominus (\mathbf{u} \wedge \mathbf{v}) == ((\mathbf{x} \wedge \mathbf{y}) \ominus \mathbf{u}) \ominus \mathbf{v} == -((\mathbf{x} \wedge \mathbf{y}) \ominus \mathbf{v}) \ominus \mathbf{u}$$

It is the straightforward and consistent derivation of formulae like [1.24] from definition [1.20] using only the fundamental exterior, regressive and complement operations, that shows how powerful Grassmann’s approach is. The alternative approach of simply *introducing* an inner product onto a space cannot bring such power to bear.

Orthogonality

As is well known, two 1-elements are said to be *orthogonal* if their scalar product is zero.

More generally, a 1-element x is orthogonal to a simple element A if and only if their interior product $A \ominus x$ is zero.

However, for $A \ominus (x_1 \wedge x_2 \wedge \cdots \wedge x_k)$ to be zero it is only necessary that *one* of the x_i be orthogonal to A. To show this, suppose it to be (without loss of generality) x_1. Then by formula [1.24] we can write

$$A \ominus (x_1 \wedge x_2 \wedge \cdots \wedge x_k) = (A \ominus x_1) \ominus (x_2 \wedge \cdots \wedge x_k)$$

Hence it becomes immediately clear that if $A \ominus x_1$ is zero then so is the whole product.

$$A \ominus x_i = 0 \quad \Longrightarrow \quad A \ominus (\ldots \wedge x_i \wedge \ldots) = 0 \qquad 1.25$$

Measure and magnitude

The *measure* of a simple element A is denoted $|A|$, and is defined to be the positive square root of the interior product of the element with itself. Suppose A is expressed as the exterior product of 1-elements, then we can use formula [1.22] as the basis for computing its *measure*.

$$A = \alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_m$$

$$|A|^2 = A \ominus A = (\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_m) \ominus (\alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_m) = \mathrm{Det}[\alpha_i \ominus \alpha_j] \qquad 1.26$$

Under a geometric interpretation of the space in which 1-elements are interpreted as vectors representing displacements, the concept of measure corresponds to the concept of *magnitude*. The magnitude of a vector is its length, the magnitude of a bivector is the area of the parallelogram formed by its two vectors, and the magnitude of a trivector is the volume of the parallelepiped formed by its three vectors. The magnitude of a scalar is the scalar itself.

The magnitude of a vector x is, as expected, given by the standard formula.

$$|x| = \sqrt{x \ominus x} \qquad 1.27$$

The magnitude of a bivector $x \wedge y$ is given by formula [1.26] as

$$|x \wedge y| = \sqrt{(x \wedge y) \ominus (x \wedge y)} = \sqrt{\mathrm{Det}\begin{bmatrix} x \ominus x & x \ominus y \\ x \ominus y & y \ominus y \end{bmatrix}} \qquad 1.28$$

Of course, a bivector may be expressed in an infinity of ways as the exterior product of two vectors, since adding a scalar multiple of the first vector to the second does not change the bivector. For example

$$B = x \wedge y = x \wedge (y + a\,x)$$

From this, the square of its area may be written in either of two ways:

$$B \ominus B = (x \wedge y) \ominus (x \wedge y)$$
$$B \ominus B = (x \wedge (y + a\,x)) \ominus (x \wedge (y + a\,x))$$

However, multiplying out these expressions using formula [1.26] shows that terms cancel in the second expression, thus reducing them both to the same expression, and demonstrating the invariance of the definition.

$$B \ominus B == (x \ominus x)\,(y \ominus y) - (x \ominus y)^2$$

Thus the measure of a bivector is independent of the actual vectors used to express it. Geometrically interpreted, this confirms the elementary result that the area of the corresponding parallelogram (with sides corresponding to the displacements represented by the vectors) is independent of its shape. These results extend straightforwardly to simple elements of any grade.

The measure of an element is equal to the measure of its complement. By the definition of the interior product [1.20], and formulae [1.18] and [1.19] we have

$$\underset{m}{\alpha} \ominus \underset{m}{\alpha} == \underset{m}{\alpha} \vee \overline{\underset{m}{\alpha}} == \overline{\overline{\underset{m}{\alpha} \vee \overline{\underset{m}{\alpha}}}} == (-1)^{m\,(n-m)}\, \overline{\overline{\underset{m}{\alpha}}} \vee \overline{\underset{m}{\alpha}} == \overline{\underset{m}{\alpha}} \vee \overline{\overline{\underset{m}{\alpha}}} == \overline{\underset{m}{\alpha}} \ominus \overline{\underset{m}{\alpha}}$$

$$A \ominus A == \overline{A} \ominus \overline{A} \qquad 1.29$$

A *unit element* $\hat{A}$ can be defined by the ratio of the element to its measure.

$$\hat{A} == \frac{A}{|A|} \qquad 1.30$$

Calculating interior products from their definition

We can use the interior product definition [1.20], the definitions of the Euclidean complement in section 1.5, and the regressive unit axiom [1.9] with $\underset{n}{1} = \overline{1} = e_1 \wedge e_2 \wedge e_3$ to calculate interior products directly from their definition. In what follows we calculate the interior products of representative basis elements of a 3-space with Euclidean metric. As expected, the scalar products $e_1 \ominus e_1$ and $e_1 \ominus e_2$ turn out to be 1 and 0 respectively.

$$e_1 \ominus e_1 == e_1 \vee \overline{e_1} == e_1 \vee (e_2 \wedge e_3) == (e_1 \wedge e_2 \wedge e_3) \vee 1 == \overline{1} \vee 1 == 1$$

$$e_1 \ominus e_2 == e_1 \vee \overline{e_2} == e_1 \vee (e_3 \wedge e_1) == (e_1 \wedge e_3 \wedge e_1) \vee 1 == 0 \vee 1 == 0$$

Using the Common Factor Axiom [1.13] with the common factor equal to 1, it is straightforward to see that inner products of identical basis 2-elements are unity.

$$(e_1 \wedge e_2) \ominus (e_1 \wedge e_2) == (e_1 \wedge e_2) \vee \overline{(e_1 \wedge e_2)} == (e_1 \wedge e_2) \vee e_3$$
$$== (e_1 \wedge e_2 \wedge e_3) \vee 1 == \overline{1} \vee 1 == 1$$

Inner products of non-identical basis 2-elements are zero.

$$(e_1 \wedge e_2) \ominus (e_2 \wedge e_3) == (e_1 \wedge e_2) \vee \overline{(e_2 \wedge e_3)} == (e_1 \wedge e_2) \vee e_1$$
$$== (e_1 \wedge e_2 \wedge e_1) \vee 1 == 0 \vee 1 == 0$$

If a basis 2-element contains a given basis 1-element, then their interior product is not zero.

$$(e_1 \wedge e_2) \ominus e_1 == (e_1 \wedge e_2) \vee \overline{e_1} == (e_1 \wedge e_2) \vee (e_2 \wedge e_3)$$
$$== (e_1 \wedge e_2 \wedge e_3) \vee e_2 == \overline{1} \vee e_2 == e_2$$

If a basis 2-element does *not* contain a given basis 1-element, then their interior product is zero:

$$(e_1 \wedge e_2) \ominus e_3 == (e_1 \wedge e_2) \vee \overline{e_3} == (e_1 \wedge e_2) \vee (e_1 \wedge e_2) == 0$$

Expanding interior products

To expand interior products, we will use the Interior Common Factor Theorem developed in Chapter 6. This theorem shows how an interior product of a simple element $\underset{m}{\alpha}$ with another, not necessarily simple element of equal or lower grade $\underset{k}{\beta}$, may be expressed as a linear combination of the ν (= $\binom{m}{k}$) essentially different factors $\underset{m-k}{\alpha_i}$ (of grade *m–k*) of the simple element of higher grade.

$$\underset{m}{\alpha} \ominus \underset{k}{\beta} == \sum_{i=1}^{\nu} \left(\underset{k}{\alpha_i} \ominus \underset{k}{\beta}\right) \underset{m-k}{\alpha_i}$$
$$\underset{m}{\alpha} == \underset{k}{\alpha_1} \wedge \underset{m-k}{\alpha_1} == \underset{k}{\alpha_2} \wedge \underset{m-k}{\alpha_2} == \cdots == \underset{k}{\alpha_\nu} \wedge \underset{m-k}{\alpha_\nu} \qquad 1.31$$

For example, the Interior Common Factor Theorem may be used to prove a relationship involving the interior product of a 1-element x with the exterior product of two factors, each of which may not be simple. This relationship and the special cases that derive from it find application throughout the algebra.

$$\left(\underset{m}{\alpha} \wedge \underset{k}{\beta}\right) \ominus x == \left(\underset{m}{\alpha} \ominus x\right) \wedge \underset{k}{\beta} + (-1)^m \underset{m}{\alpha} \wedge \left(\underset{k}{\beta} \ominus x\right) \qquad 1.32$$

The Interior Common Factor Theorem may also be expressed in a more algorithmically powerful form in terms of the span and cospan of $\underset{m}{\alpha}$.

The interior product of a bivector and a vector

Suppose x is a vector and $B = x_1 \wedge x_2$ is a simple bivector. The interior product of the bivector B with the vector x is the vector $B \ominus x$. This can be expanded by the Interior Common Factor Theorem, or formula [1.32] to give:

$$B \ominus x == (x_1 \wedge x_2) \ominus x == (x \ominus x_1)\, x_2 - (x \ominus x_2)\, x_1$$

Since $B \ominus x$ is expressed as a linear combination of x_1 and x_2 it is clearly *contained in* the bivector B so that the exterior product of B with $B \ominus x$ is zero.

$$B \wedge (B \ominus x) == 0$$

The resulting vector $B \ominus x$ is also *orthogonal* to x. We can show this by taking its scalar product with x, and then using formula [1.24].

$$(B \ominus x) \ominus x == B \ominus (x \wedge x) == 0$$

If $\hat{B}$ is the unit bivector of B, the projection $x^{\#}$ of x onto B is given by

$$x^{\#} == -\hat{B} \ominus \left(\hat{B} \ominus x\right)$$

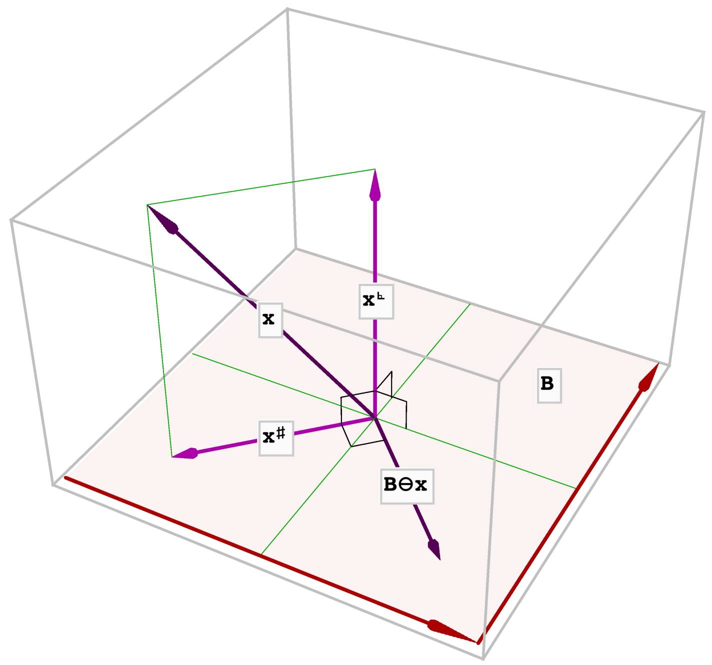
The component $x^{\perp}$ of x orthogonal to B is given by

$$x^{\perp} == \left(\hat{B} \wedge x\right) \ominus \hat{B}$$

It is easily shown that the sum of these two components is equal to x.

$$x == x^{\#} + x^{\perp}$$

In the diagram below we depict these relationships. But remember that the vectors and bivector are only docked in a *convenient* location.



The interior product of a bivector with a vector

These concepts may easily be extended to geometric entities of higher grade. We will explore them further in Chapter 6.

The cross product

The cross or vector product of the three-dimensional vector calculus of Gibbs *et al*. [Gibbs 1928] corresponds to two operations in Grassmann's more general calculus. *Taking the cross-product of two vectors in three dimensions corresponds to taking the complement of their exterior product*. However, whilst the usual cross product formulation is valid only for vectors in three dimensions, the exterior product formulation is valid for elements of *any* grade in *any* number of dimensions. Therefore the opportunity exists to generalize the concept.

Because our generalization reduces to the usual definition under the usual circumstances, we take the liberty of continuing to refer to the generalized cross product as, simply, the cross product.

Let A and B be elements of any grade, then their cross product is denoted $A \times B$ and is defined as the complement of their exterior product. The cross product of an *m*-element and a *k*-element is thus an $(n-(m+k))$-element.

$$A \times B = \overline{A \wedge B} \qquad 1.33$$

This definition preserves the basic property of the cross product: that the cross product of two elements is an element orthogonal to both, and reduces to the usual notion for vectors in a three dimensional metric vector space. For 1-elements x_i the definition has the following consequences, independent of the dimension of the space.

- The triple cross product is a 1-element in any number of dimensions.

$$(x_1 \times x_2) \times x_3 = (x_1 \wedge x_2) \ominus x_3 = (x_3 \ominus x_1)\, x_2 - (x_3 \ominus x_2)\, x_1 \qquad 1.34$$

- The box product, or scalar triple product, is an (n–3)-element, and therefore a scalar only in three dimensions.

$$(x_1 \times x_2) \ominus x_3 = \overline{x_1 \wedge x_2 \wedge x_3} \qquad 1.35$$

- The scalar product of two cross products is a scalar in any number of dimensions.

$$(x_1 \times x_2) \ominus (x_3 \times x_4) = (x_1 \wedge x_2) \ominus (x_3 \wedge x_4) \qquad 1.36$$

- The cross product of two cross products is a (4–n)-element, and therefore a 1-element only in three dimensions. It corresponds to the regressive product of two exterior products.

$$(x_1 \times x_2) \times (x_3 \times x_4) = (x_1 \wedge x_2) \vee (x_3 \wedge x_4) \qquad 1.37$$

The cross product of the three-dimensional vector calculus requires the space to have a metric, since it is defined to be orthogonal to its factors. Equation [1.37] however, shows that in this particular case, the result does not explicitly require a metric.

1.7 Summary

Summary of operations

In this chapter we have briefly introduced the four fundamental operations which underpin the Grassmann algebra: the exterior product, the regressive product, the complement operation, and the interior product. They have been introduced in this sequence because each one depends on those preceding.

The *exterior product* is the first of the fundamental operations upon which all the others are based. It encodes the notion of linear independence in a way that enables higher order entities (for example, lines and planes) to be constructed from lower order ones (for example, points and vectors).

The *regressive product* was introduced as a true dual product operation to the exterior product. It is a true dual because its axioms can be derived from the axioms of the exterior product and *vice-versa*. The intersection of two lines in the plane was computed using the regressive product, previewing how it will be particularly powerful in Projective Geometry to compute intersections of any geometric entities in any space.

The exterior and regressive products, although they can build entities and intersect them, cannot measure or compare any of them unless they are congruent. For example, given two vectors which are not scalar multiples of each other, the two products do not lead to any invariant mechanism for deciding which is ‘larger’. To do this we need to provide the algebra with more information with which to make its decisions. The complement operation introduces just such information.

The *complement* operation is a rule (or mapping) which, given an element, A say, of the algebra enables us to correspond an element $\overline{A}$, such that the exterior product of A with $\overline{A}$ is equal to a scalar multiple of the unit *n*-element of the space; and the regressive product of A with $\overline{A}$ is equal simply to this scalar multiple. If we take this scalar to be the square of a quantity which we call the *measure* or *magnitude* of A, we find a beautiful correspondence between magnitudes computed thus, and the commonly accepted magnitudes associated with geometric figures. For example the magnitude of a vector corresponds to its length, the magnitude of a bivector corresponds to the area of any of the parallelograms formed from its vectors, and the magnitude of a trivector corresponds to the volume of any of the parallelepipeds formed from its vectors.

The complement mapping can be defined on the elements of any basis chosen for the underlying linear space of the algebra and induced onto the rest of the algebra. That is, once we can measure basis elements, we can measure any entity of the algebra. This ability to measure in a way which conforms to our notions of physical measurement of geometric objects is so important that this scalar square of the magnitude of the element A is defined to be a new product: the *inner product* of A with itself - defined as the regressive product of A with its complement. It is straightforward then to extend this to general elements by defining the *interior product* of A with B as the regressive product of A with the complement of B.

Summary of objects

In the previous section we have discussed the fundamental *operations* of Grassmann algebra. However, an important contributor to the power of the algebra is that its *objects* may be endowed with different geometric interpretations.

0. A *multilinear algebra* like the Grassmann algebra is just the algebra you need when you have more than one independent object. Like all mathematics, it is devoid of interpretation, existing only as a set of objects, and a consistent set of operations and rules.

1. Interpreting all the basic objects of a multilinear algebra with just the exterior and regressive products as 'directed line segments with no location' gives us *vector geometry*.

2. If to vector geometry we add another object, and interpret it as the 'origin point', we get *projective geometry*.

3. If to vector geometry we add a rule for measuring and comparing the objects we get *metric vector geometry*.

4. If to projective geometry we add a rule for measuring and comparing the objects we get *metric projective geometry*.

5. Some physical phenomena can be modelled by one or more of these types of geometry by representing the physical 'objects' by geometric ones.